1 Russell and Whitehead

1.1 Definitions

or $(A \vee B)$ expanded to $(A \vee B)$
not $(\neg A)$ expanded to $(\neg A)$
impl $((\neg A) \vee B)$ expanded to $((\neg A) \vee B)$
and $(\neg((\neg A) \vee (\neg B)))$ expanded to $(\neg((\neg A) \vee (\neg B)))$
equ $((A \rightarrow B) \wedge (B \rightarrow A))$ expanded to $(\neg((\neg((\neg A) \vee B)) \vee (\neg((\neg B) \vee A))))$

1.2 Axioms

axiom1 $((A \vee A) \rightarrow A)$ expanded to $((\neg(A \vee A)) \vee A)$
axiom2 $(A \rightarrow (B \vee A))$ expanded to $((\neg A) \vee (B \vee A))$
axiom3 $((A \vee B) \rightarrow (B \vee A))$ expanded to $((\neg(A \vee B)) \vee (B \vee A))$
axiom4 $((A \vee (B \vee C)) \rightarrow (B \vee (A \vee C)))$ expanded to $((\neg(A \vee (B \vee C))) \vee (B \vee (A \vee C)))$
axiom5 $((A \rightarrow B) \rightarrow ((C \vee A) \rightarrow (C \vee B)))$ expanded to $((\neg((\neg A) \vee B)) \vee ((\neg(C \vee A)) \vee (C \vee B)))$

1.3 Theorems

theorem1: $((a \rightarrow (\neg a)) \rightarrow (\neg a))$ **expanded to** $((\neg((\neg a) \vee (\neg a))) \vee (\neg a))$
1. $((\neg(A \vee A)) \vee A)$ add proposition axiom1
2. $((\neg((\neg a) \vee (\neg a))) \vee (\neg a))$ substitute A with $(\neg a)$ in 1

theorem2: $(a \rightarrow (a \vee b))$ **expanded to** $((\neg a) \vee (a \vee b))$
1. $((\neg((\neg A) \vee B)) \vee ((\neg(C \vee A)) \vee (C \vee B)))$ add proposition axiom5
2. $((\neg((\neg A) \vee (a \vee b))) \vee ((\neg(C \vee A)) \vee (C \vee (a \vee b))))$ substitute B with $(a \vee b)$ in 1
3. $((\neg((\neg A) \vee (a \vee b))) \vee ((\neg((\neg a) \vee A)) \vee ((\neg a) \vee (a \vee b))))$ substitute C with $(\neg a)$ in 2
4. $((\neg((\neg(b \vee a)) \vee (a \vee b))) \vee ((\neg((\neg a) \vee (b \vee a))) \vee ((\neg a) \vee (a \vee b))))$ substitute A with $(b \vee a)$ in 3
5. $((\neg(A \vee B)) \vee (B \vee A))$ add proposition axiom3
6. $((\neg(b \vee B)) \vee (B \vee b))$ substitute A with b in 5
7. $((\neg(b \vee a)) \vee (a \vee b))$ substitute B with a in 6
8. $((\neg((\neg a) \vee (b \vee a))) \vee ((\neg a) \vee (a \vee b)))$ modus ponens 4 and 7
9. $((\neg A) \vee (B \vee A))$ add proposition axiom2
10. $((\neg a) \vee (B \vee a))$ substitute A with a in 9
11. $((\neg a) \vee (b \vee a))$ substitute B with b in 10
12. $((\neg a) \vee (a \vee b))$ modus ponens 8 and 11
13. $((\neg a) \vee (a \vee b))$ initial proposition

theorem3: $(a \rightarrow (a \vee a))$ **expanded to** $((\neg a) \vee (a \vee a))$
1. $((\neg A) \vee (B \vee A))$ add proposition axiom2
2. $((\neg a) \vee (B \vee a))$ substitute A with a in 1

3. $((\neg a) \vee (a \vee a))$ substitute B with a in 2

theorem4: $(a \rightarrow (\neg(\neg a)))$ **expanded to** $((\neg a) \vee (\neg(\neg a)))$

1. $((\neg((\neg A) \vee B)) \vee ((\neg(C \vee A)) \vee (C \vee B)))$ add proposition axiom5
2. $((\neg((\neg A) \vee (\neg(\neg a)))) \vee ((\neg(C \vee A)) \vee (C \vee (\neg(\neg a)))))$ substitute B with $(\neg(\neg a))$ in 1
3. $((\neg((\neg A) \vee (\neg(\neg a)))) \vee ((\neg((\neg a) \vee A)) \vee ((\neg a) \vee (\neg(\neg a)))))$ substitute C with $(\neg a)$ in 2
4. $((\neg((\neg((\neg(\neg a)) \vee (\neg(\neg a)))) \vee (\neg(\neg a)))) \vee ((\neg((\neg a) \vee ((\neg(\neg a)) \vee (\neg(\neg a))))) \vee ((\neg a) \vee (\neg(\neg a)))))$ substitute A with $((\neg(\neg a)) \vee (\neg(\neg a)))$ in 3
5. $((\neg(A \vee A)) \vee A)$ add proposition axiom1
6. $((\neg((\neg(\neg a)) \vee (\neg(\neg a)))) \vee (\neg(\neg a)))$ substitute A with $(\neg(\neg a))$ in 5
7. $((\neg((\neg a) \vee ((\neg(\neg a)) \vee (\neg(\neg a))))) \vee ((\neg a) \vee (\neg(\neg a))))$ modus ponens 4 and 6
8. $((\neg(A \vee (B \vee C))) \vee (B \vee (A \vee C)))$ add proposition axiom4
9. $((\neg((\neg(\neg a)) \vee (B \vee C))) \vee (B \vee ((\neg(\neg a)) \vee C)))$ substitute A with $(\neg(\neg a))$ in 8
10. $((\neg((\neg(\neg a)) \vee ((\neg a) \vee C))) \vee ((\neg a) \vee ((\neg(\neg a)) \vee C)))$ substitute B with $(\neg a)$ in 9
11. $((\neg((\neg(\neg a)) \vee ((\neg a) \vee (\neg(\neg a))))) \vee ((\neg a) \vee ((\neg(\neg a)) \vee (\neg(\neg a)))))$ substitute C with $(\neg(\neg a))$ in 10
12. $((\neg A) \vee (A \vee B))$ add proposition theorem2
13. $((\neg(\neg a)) \vee ((\neg a) \vee B))$ substitute A with $(\neg a)$ in 12
14. $((\neg(\neg a)) \vee ((\neg a) \vee (\neg(\neg a))))$ substitute B with $(\neg(\neg a))$ in 13
15. $((\neg a) \vee ((\neg(\neg a)) \vee (\neg(\neg a))))$ modus ponens 11 and 14
16. $((\neg a) \vee (\neg(\neg a)))$ modus ponens 7 and 15
17. $((\neg a) \vee (\neg(\neg a)))$ initial proposition

theorem5: $((\neg(a \vee b)) \rightarrow (\neg a))$ **expanded to** $((\neg(\neg(a \vee b))) \vee (\neg a))$

1. $((\neg(A \vee B)) \vee (B \vee A))$ add proposition axiom3
2. $((\neg((\neg a) \vee B)) \vee (B \vee (\neg a)))$ substitute A with $(\neg a)$ in 1
3. $((\neg((\neg a) \vee (\neg(\neg(a \vee b))))) \vee ((\neg(\neg(a \vee b))) \vee (\neg a)))$ substitute B with $(\neg(\neg(a \vee b)))$ in 2
4. $((\neg((\neg A) \vee B)) \vee ((\neg(C \vee A)) \vee (C \vee B)))$ add proposition axiom5
5. $((\neg((\neg A) \vee (\neg(\neg(a \vee b))))) \vee ((\neg(C \vee A)) \vee (C \vee (\neg(\neg(a \vee b))))))$ substitute B with $(\neg(\neg(a \vee b)))$ in 4
6. $((\neg((\neg A) \vee (\neg(\neg(a \vee b))))) \vee ((\neg((\neg a) \vee A)) \vee ((\neg a) \vee (\neg(\neg(a \vee b))))))$ substitute C with $(\neg a)$ in 5
7. $((\neg((\neg(a \vee b)) \vee (\neg(\neg(a \vee b))))) \vee ((\neg((\neg a) \vee (a \vee b))) \vee ((\neg a) \vee (\neg(\neg(a \vee b))))))$ substitute A with $(a \vee b)$ in 6
8. $((\neg A) \vee (\neg(\neg A)))$ add proposition theorem4
9. $((\neg(a \vee b)) \vee (\neg(\neg(a \vee b))))$ substitute A with $(a \vee b)$ in 8
10. $((\neg((\neg a) \vee (a \vee b))) \vee ((\neg a) \vee (\neg(\neg(a \vee b)))))$ modus ponens 7 and 9
11. $((\neg A) \vee (A \vee B))$ add proposition theorem2
12. $((\neg a) \vee (a \vee B))$ substitute A with a in 11
13. $((\neg a) \vee (a \vee b))$ substitute B with b in 12
14. $((\neg a) \vee (\neg(\neg(a \vee b))))$ modus ponens 10 and 13
15. $((\neg(\neg(a \vee b))) \vee (\neg a))$ modus ponens 3 and 14
16. $((\neg(\neg(a \vee b))) \vee (\neg a))$ initial proposition

2 Hilbert and Ackermann

2.1 Definitions

or $(A \vee B)$ expanded to $(A \vee B)$
not $(\neg A)$ expanded to $(\neg A)$
impl $((\neg A) \vee B)$ expanded to $((\neg A) \vee B)$
and $(\neg((\neg A) \vee (\neg B)))$ expanded to $(\neg((\neg A) \vee (\neg B)))$
equ $((A \rightarrow B) \wedge (B \rightarrow A))$ expanded to $(\neg((\neg((\neg A) \vee B)) \vee (\neg((\neg B) \vee A))))$

2.2 Axioms

axiom1 $((A \vee A) \rightarrow A)$ expanded to $((\neg(A \vee A)) \vee A)$
axiom2 $(A \rightarrow (A \vee B))$ expanded to $((\neg A) \vee (A \vee B))$
axiom3 $((A \vee B) \rightarrow (B \vee A))$ expanded to $((\neg(A \vee B)) \vee (B \vee A))$
axiom4 $((A \rightarrow B) \rightarrow ((C \vee A) \rightarrow (C \vee B)))$ expanded to $((\neg((\neg A) \vee B)) \vee ((\neg(C \vee A)) \vee (C \vee B)))$

2.3 Theorems

theorem1: $((a \rightarrow b) \rightarrow ((c \rightarrow a) \rightarrow (c \rightarrow b)))$ **expanded to** $((\neg((\neg a) \vee b)) \vee ((\neg((\neg c) \vee a)) \vee ((\neg c) \vee b)))$
1. $((\neg((\neg A) \vee B)) \vee ((\neg(C \vee A)) \vee (C \vee B)))$ add proposition axiom4
2. $((\neg((\neg a) \vee B)) \vee ((\neg(C \vee a)) \vee (C \vee B)))$ substitute A with a in 1
3. $((\neg((\neg a) \vee b)) \vee ((\neg(C \vee a)) \vee (C \vee b)))$ substitute B with b in 2
4. $((\neg((\neg a) \vee b)) \vee ((\neg((\neg c) \vee a)) \vee ((\neg c) \vee b)))$ substitute C with $(\neg c)$ in 3

theorem2: $(a \rightarrow a)$ **expanded to** $((\neg a) \vee a)$
1. $((\neg((\neg A) \vee B)) \vee ((\neg(C \vee A)) \vee (C \vee B)))$ add proposition axiom4
2. $((\neg((\neg A) \vee a)) \vee ((\neg(C \vee A)) \vee (C \vee a)))$ substitute B with a in 1
3. $((\neg((\neg A) \vee a)) \vee ((\neg((\neg a) \vee A)) \vee ((\neg a) \vee a)))$ substitute C with $(\neg a)$ in 2
4. $((\neg((\neg(a \vee a)) \vee a)) \vee ((\neg((\neg a) \vee (a \vee a))) \vee ((\neg a) \vee a)))$ substitute A with $(a \vee a)$ in 3
5. $((\neg(A \vee A)) \vee A)$ add proposition axiom1
6. $((\neg(a \vee a)) \vee a)$ substitute A with a in 5
7. $((\neg((\neg a) \vee (a \vee a))) \vee ((\neg a) \vee a))$ modus ponens 4 and 6
8. $((\neg A) \vee (A \vee B))$ add proposition axiom2
9. $((\neg a) \vee (a \vee B))$ substitute A with a in 8
10. $((\neg a) \vee (a \vee a))$ substitute B with a in 9
11. $((\neg a) \vee a)$ modus ponens 7 and 10
12. $((\neg a) \vee a)$ initial proposition

theorem3: $((\neg a) \vee a)$ **expanded to** $((\neg a) \vee a)$
1. $((\neg A) \vee A)$ add proposition theorem2
2. $((\neg a) \vee a)$ substitute A with a in 1

theorem4: $(a \vee (\neg a))$ **expanded to** $(a \vee (\neg a))$
1. $((\neg(A \vee B)) \vee (B \vee A))$ add proposition axiom3

2. $((\neg((\neg a) \vee B)) \vee (B \vee (\neg a)))$ substitute A with $(\neg a)$ in 1
3. $((\neg((\neg a) \vee a)) \vee (a \vee (\neg a)))$ substitute B with a in 2
4. $((\neg A) \vee A)$ add proposition theorem2
5. $((\neg a) \vee a)$ substitute A with a in 4
6. $(a \vee (\neg a))$ modus ponens 3 and 5
7. $(a \vee (\neg a))$ initial proposition

theorem5: $(a \rightarrow (\neg(\neg a)))$ **expanded to** $((\neg a) \vee (\neg(\neg a)))$
1. $(A \vee (\neg A))$ add proposition theorem4
2. $((\neg a) \vee (\neg(\neg a)))$ substitute A with $(\neg a)$ in 1

theorem6: $((\neg(\neg a)) \rightarrow a)$ **expanded to** $((\neg(\neg(\neg a))) \vee a)$
1. $((\neg(A \vee B)) \vee (B \vee A))$ add proposition axiom3
2. $((\neg(a \vee B)) \vee (B \vee a))$ substitute A with a in 1
3. $((\neg(a \vee (\neg(\neg(\neg a))))) \vee ((\neg(\neg(\neg a))) \vee a))$ substitute B with $(\neg(\neg(\neg a)))$ in 2
4. $((\neg((\neg A) \vee B)) \vee ((\neg(C \vee A)) \vee (C \vee B)))$ add proposition axiom4
5. $((\neg((\neg A) \vee (\neg(\neg(\neg a))))) \vee ((\neg(C \vee A)) \vee (C \vee (\neg(\neg(\neg a))))))$ substitute B with $(\neg(\neg(\neg a)))$ in 4
6. $((\neg((\neg A) \vee (\neg(\neg(\neg a))))) \vee ((\neg(a \vee A)) \vee (a \vee (\neg(\neg(\neg a))))))$ substitute C with a in 5
7. $((\neg((\neg(\neg a)) \vee (\neg(\neg(\neg a))))) \vee ((\neg(a \vee (\neg a))) \vee (a \vee (\neg(\neg(\neg a))))))$ substitute A with $(\neg a)$ in 6
8. $(A \vee (\neg A))$ add proposition theorem4
9. $((\neg(\neg a)) \vee (\neg(\neg(\neg a))))$ substitute A with $(\neg(\neg a))$ in 8
10. $((\neg(a \vee (\neg a))) \vee (a \vee (\neg(\neg(\neg a)))))$ modus ponens 7 and 9
11. $(A \vee (\neg A))$ add proposition theorem4
12. $(a \vee (\neg a))$ substitute A with a in 11
13. $(a \vee (\neg(\neg(\neg a))))$ modus ponens 10 and 12
14. $((\neg(\neg(\neg a))) \vee a)$ modus ponens 3 and 13
15. $((\neg(\neg(\neg a))) \vee a)$ initial proposition

theorem7: $((a \rightarrow b) \rightarrow ((\neg b) \rightarrow (\neg a)))$ **expanded to** $((\neg((\neg a) \vee b)) \vee ((\neg(\neg b)) \vee (\neg a)))$
1. $((\neg((\neg A) \vee B)) \vee ((\neg(C \vee A)) \vee (C \vee B)))$ add proposition axiom4
2. $((\neg((\neg A) \vee ((\neg(\neg b)) \vee (\neg a)))) \vee ((\neg(C \vee A)) \vee (C \vee ((\neg(\neg b)) \vee (\neg a)))))$ substitute B with $((\neg(\neg b)) \vee (\neg a))$ in 1
3. $((\neg((\neg A) \vee ((\neg(\neg b)) \vee (\neg a)))) \vee ((\neg((\neg((\neg a) \vee b)) \vee A)) \vee ((\neg((\neg a) \vee b)) \vee ((\neg(\neg b)) \vee (\neg a)))))$ substitute C with $(\neg((\neg a) \vee b))$ in 2
4. $((\neg((\neg((\neg a) \vee (\neg(\neg b)))) \vee ((\neg(\neg b)) \vee (\neg a)))) \vee ((\neg((\neg((\neg a) \vee b)) \vee ((\neg a) \vee (\neg(\neg b))))) \vee ((\neg((\neg a) \vee b)) \vee ((\neg(\neg b)) \vee (\neg a)))))$ substitute A with $((\neg a) \vee (\neg(\neg b)))$ in 3
5. $((\neg(A \vee B)) \vee (B \vee A))$ add proposition axiom3
6. $((\neg((\neg a) \vee B)) \vee (B \vee (\neg a)))$ substitute A with $(\neg a)$ in 5
7. $((\neg((\neg a) \vee (\neg(\neg b)))) \vee ((\neg(\neg b)) \vee (\neg a)))$ substitute B with $(\neg(\neg b))$ in 6
8. $((\neg((\neg((\neg a) \vee b)) \vee ((\neg a) \vee (\neg(\neg b))))) \vee ((\neg((\neg a) \vee b)) \vee ((\neg(\neg b)) \vee (\neg a))))$ modus ponens 4 and 7
9. $((\neg((\neg A) \vee B)) \vee ((\neg(C \vee A)) \vee (C \vee B)))$ add proposition axiom4
10. $((\neg((\neg b) \vee B)) \vee ((\neg(C \vee b)) \vee (C \vee B)))$ substitute A with b in 9
11. $((\neg((\neg b) \vee (\neg(\neg b)))) \vee ((\neg(C \vee b)) \vee (C \vee (\neg(\neg b)))))$ substitute B with $(\neg(\neg b))$ in 10

12. $((\neg((\neg b) \vee (\neg(\neg b)))) \vee ((\neg((\neg a) \vee b)) \vee ((\neg a) \vee (\neg(\neg b)))))$ substitute C with $(\neg a)$ in 11
13. $(A \vee (\neg A))$ add proposition theorem4
14. $((\neg b) \vee (\neg(\neg b)))$ substitute A with $(\neg b)$ in 13
15. $((\neg((\neg a) \vee b)) \vee ((\neg a) \vee (\neg(\neg b))))$ modus ponens 12 and 14
16. $((\neg((\neg a) \vee b)) \vee ((\neg(\neg b)) \vee (\neg a)))$ modus ponens 8 and 15
17. $((\neg((\neg a) \vee b)) \vee ((\neg(\neg b)) \vee (\neg a)))$ initial proposition

theorem8: $(a \rightarrow (b \vee a))$ **expanded to** $((\neg a) \vee (b \vee a))$
1. $((\neg((\neg A) \vee B)) \vee ((\neg(C \vee A)) \vee (C \vee B)))$ add proposition axiom4
2. $((\neg((\neg A) \vee (b \vee a))) \vee ((\neg(C \vee A)) \vee (C \vee (b \vee a))))$ substitute B with $(b \vee a)$ in 1
3. $((\neg((\neg A) \vee (b \vee a))) \vee ((\neg((\neg a) \vee A)) \vee ((\neg a) \vee (b \vee a))))$ substitute C with $(\neg a)$ in 2
4. $((\neg((\neg(a \vee b)) \vee (b \vee a))) \vee ((\neg((\neg a) \vee (a \vee b))) \vee ((\neg a) \vee (b \vee a))))$ substitute A with $(a \vee b)$ in 3
5. $((\neg(A \vee B)) \vee (B \vee A))$ add proposition axiom3
6. $((\neg(a \vee B)) \vee (B \vee a))$ substitute A with a in 5
7. $((\neg(a \vee b)) \vee (b \vee a))$ substitute B with b in 6
8. $((\neg((\neg a) \vee (a \vee b))) \vee ((\neg a) \vee (b \vee a)))$ modus ponens 4 and 7
9. $((\neg A) \vee (A \vee B))$ add proposition axiom2
10. $((\neg a) \vee (a \vee B))$ substitute A with a in 9
11. $((\neg a) \vee (a \vee b))$ substitute B with b in 10
12. $((\neg a) \vee (b \vee a))$ modus ponens 8 and 11
13. $((\neg a) \vee (b \vee a))$ initial proposition

theorem9: $((a \rightarrow (\neg a)) \rightarrow (\neg a))$ **expanded to** $((\neg((\neg a) \vee (\neg a))) \vee (\neg a))$
1. $((\neg(A \vee A)) \vee A)$ add proposition axiom1
2. $((\neg((\neg a) \vee (\neg a))) \vee (\neg a))$ substitute A with $(\neg a)$ in 1

theorem10: $(a \rightarrow (a \vee b))$ **expanded to** $((\neg a) \vee (a \vee b))$
1. $((\neg A) \vee (A \vee B))$ add proposition axiom2
2. $((\neg a) \vee (a \vee B))$ substitute A with a in 1
3. $((\neg a) \vee (a \vee b))$ substitute B with b in 2

theorem11: $(a \rightarrow (a \vee a))$ **expanded to** $((\neg a) \vee (a \vee a))$
1. $((\neg A) \vee (A \vee B))$ add proposition axiom2
2. $((\neg a) \vee (a \vee B))$ substitute A with a in 1
3. $((\neg a) \vee (a \vee a))$ substitute B with a in 2

theorem12: $(a \rightarrow (\neg(\neg a)))$ **expanded to** $((\neg a) \vee (\neg(\neg a)))$
1. $(A \vee (\neg A))$ add proposition theorem4
2. $((\neg a) \vee (\neg(\neg a)))$ substitute A with $(\neg a)$ in 1

theorem13: $(((\neg a) \rightarrow b) \rightarrow ((\neg b) \rightarrow a))$ **expanded to** $((\neg((\neg(\neg a)) \vee b)) \vee ((\neg(\neg b)) \vee a))$
1. $((\neg((\neg A) \vee B)) \vee ((\neg(C \vee A)) \vee (C \vee B)))$ add proposition axiom4
2. $((\neg((\neg A) \vee ((\neg(\neg b)) \vee a))) \vee ((\neg(C \vee A)) \vee (C \vee ((\neg(\neg b)) \vee a))))$ substitute B with $((\neg(\neg b)) \vee a)$ in 1
3. $((\neg((\neg A) \vee ((\neg(\neg b)) \vee a))) \vee ((\neg((\neg((\neg(\neg a)) \vee b)) \vee A)) \vee ((\neg((\neg(\neg a)) \vee b)) \vee$

$((\neg(\neg b)) \vee a))))$ substitute C with $(\neg((\neg(\neg a)) \vee b))$ in 2
4. $((\neg((\neg((\neg(\neg b))\vee(\neg(\neg a))))\vee((\neg(\neg b))\vee a)))\vee((\neg((\neg((\neg(\neg a))\vee b))\vee((\neg(\neg b))\vee(\neg(\neg a)))))\vee((\neg((\neg(\neg a))\vee b))\vee((\neg(\neg b))\vee a))))$ substitute A with $((\neg(\neg b))\vee(\neg(\neg a)))$ in 3
5. $((\neg((\neg A)\vee B))\vee((\neg(C\vee A))\vee(C\vee B)))$ add proposition axiom4
6. $((\neg((\neg(\neg(\neg a)))\vee B))\vee((\neg(C\vee(\neg(\neg a))))\vee(C\vee B)))$ substitute A with $(\neg(\neg a))$ in 5
7. $((\neg((\neg(\neg(\neg a)))\vee a))\vee((\neg(C\vee(\neg(\neg a))))\vee(C\vee a)))$ substitute B with a in 6
8. $((\neg((\neg(\neg(\neg a)))\vee a))\vee((\neg((\neg(\neg b))\vee(\neg(\neg a))))\vee((\neg(\neg b))\vee a)))$ substitute C with $(\neg(\neg b))$ in 7
9. $((\neg(\neg(\neg A)))\vee A)$ add proposition theorem6
10. $((\neg(\neg(\neg a)))\vee a)$ substitute A with a in 9
11. $((\neg((\neg(\neg b))\vee(\neg(\neg a))))\vee((\neg(\neg b))\vee a))$ modus ponens 8 and 10
12. $((\neg((\neg((\neg(\neg a))\vee b))\vee((\neg(\neg b))\vee(\neg(\neg a)))))\vee((\neg((\neg(\neg a))\vee b))\vee((\neg(\neg b))\vee a)))$ modus ponens 4 and 11
13. $((\neg((\neg A)\vee B))\vee((\neg(\neg B))\vee(\neg A)))$ add proposition theorem7
14. $((\neg((\neg(\neg a))\vee B))\vee((\neg(\neg B))\vee(\neg(\neg a))))$ substitute A with $(\neg a)$ in 13
15. $((\neg((\neg(\neg a))\vee b))\vee((\neg(\neg b))\vee(\neg(\neg a))))$ substitute B with b in 14
16. $((\neg((\neg(\neg a))\vee b))\vee((\neg(\neg b))\vee a))$ modus ponens 12 and 15
17. $((\neg((\neg(\neg a))\vee b))\vee((\neg(\neg b))\vee a))$ initial proposition

theorem14: $((a \to b) \to ((\neg b) \to (\neg a)))$ **expanded to** $((\neg((\neg a)\vee b))\vee((\neg(\neg b))\vee(\neg a)))$
1. $((\neg((\neg A)\vee B))\vee((\neg(\neg B))\vee(\neg A)))$ add proposition theorem7
2. $((\neg((\neg a)\vee B))\vee((\neg(\neg B))\vee(\neg a)))$ substitute A with a in 1
3. $((\neg((\neg a)\vee b))\vee((\neg(\neg b))\vee(\neg a)))$ substitute B with b in 2

theorem15: $(((\neg a) \to (\neg b)) \to (b \to a))$ **expanded to** $((\neg((\neg(\neg a))\vee(\neg b)))\vee((\neg b)\vee a))$
1. $((\neg((\neg A)\vee B))\vee((\neg(C\vee A))\vee(C\vee B)))$ add proposition axiom4
2. $((\neg((\neg A)\vee((\neg b)\vee a)))\vee((\neg(C\vee A))\vee(C\vee((\neg b)\vee a))))$ substitute B with $((\neg b)\vee a)$ in 1
3. $((\neg((\neg A)\vee((\neg b)\vee a)))\vee((\neg((\neg((\neg(\neg a))\vee(\neg b)))\vee A))\vee((\neg((\neg(\neg a))\vee(\neg b)))\vee((\neg b)\vee a))))$ substitute C with $(\neg((\neg(\neg a))\vee(\neg b)))$ in 2
4. $((\neg((\neg((\neg b)\vee(\neg(\neg a))))\vee((\neg b)\vee a)))\vee((\neg((\neg((\neg(\neg a))\vee(\neg b)))\vee((\neg b)\vee(\neg(\neg a)))))\vee((\neg((\neg(\neg a))\vee(\neg b)))\vee((\neg b)\vee a))))$ substitute A with $((\neg b)\vee(\neg(\neg a)))$ in 3
5. $((\neg((\neg A)\vee B))\vee((\neg(C\vee A))\vee(C\vee B)))$ add proposition axiom4
6. $((\neg((\neg(\neg(\neg a)))\vee B))\vee((\neg(C\vee(\neg(\neg a))))\vee(C\vee B)))$ substitute A with $(\neg(\neg a))$ in 5
7. $((\neg((\neg(\neg(\neg a)))\vee a))\vee((\neg(C\vee(\neg(\neg a))))\vee(C\vee a)))$ substitute B with a in 6
8. $((\neg((\neg(\neg(\neg a)))\vee a))\vee((\neg((\neg b)\vee(\neg(\neg a))))\vee((\neg b)\vee a)))$ substitute C with $(\neg b)$ in 7
9. $((\neg(\neg(\neg A)))\vee A)$ add proposition theorem6
10. $((\neg(\neg(\neg a)))\vee a)$ substitute A with a in 9
11. $((\neg((\neg b)\vee(\neg(\neg a))))\vee((\neg b)\vee a))$ modus ponens 8 and 10
12. $((\neg((\neg((\neg(\neg a))\vee(\neg b)))\vee((\neg b)\vee(\neg(\neg a)))))\vee((\neg((\neg(\neg a))\vee(\neg b)))\vee((\neg b)\vee a)))$ modus ponens 4 and 11
13. $((\neg(A\vee B))\vee(B\vee A))$ add proposition axiom3
14. $((\neg((\neg(\neg a))\vee B))\vee(B\vee(\neg(\neg a))))$ substitute A with $(\neg(\neg a))$ in 13

15. $((\neg((\neg(\neg a)) \vee (\neg b))) \vee ((\neg b) \vee (\neg(\neg a))))$ substitute B with $(\neg b)$ in 14
16. $((\neg((\neg(\neg a)) \vee (\neg b))) \vee ((\neg b) \vee a))$ modus ponens 12 and 15
17. $((\neg((\neg(\neg a)) \vee (\neg b))) \vee ((\neg b) \vee a))$ initial proposition

theorem16: $((\neg(a \vee b)) \rightarrow (\neg a))$ **expanded to** $((\neg(\neg(a \vee b))) \vee (\neg a))$
1. $((\neg((\neg A) \vee B)) \vee ((\neg(\neg B)) \vee (\neg A)))$ add proposition theorem7
2. $((\neg((\neg a) \vee B)) \vee ((\neg(\neg B)) \vee (\neg a)))$ substitute A with a in 1
3. $((\neg((\neg a) \vee (a \vee b))) \vee ((\neg(\neg(a \vee b))) \vee (\neg a)))$ substitute B with $(a \vee b)$ in 2
4. $((\neg A) \vee (A \vee B))$ add proposition axiom2
5. $((\neg a) \vee (a \vee B))$ substitute A with a in 4
6. $((\neg a) \vee (a \vee b))$ substitute B with b in 5
7. $((\neg(\neg(a \vee b))) \vee (\neg a))$ modus ponens 3 and 6
8. $((\neg(\neg(a \vee b))) \vee (\neg a))$ initial proposition

3 Mendelson

3.1 Definitions

impl $(A \rightarrow B)$ expanded to $(A \rightarrow B)$
not $(\neg A)$ expanded to $(\neg A)$
or $((\neg A) \rightarrow B)$ expanded to $((\neg A) \rightarrow B)$
and $(\neg(A \rightarrow (\neg B)))$ expanded to $(\neg(A \rightarrow (\neg B)))$
equ $((A \rightarrow B) \wedge (B \rightarrow A))$ expanded to $(\neg((A \rightarrow B) \rightarrow (\neg(B \rightarrow A))))$

3.2 Axioms

axiom1 $(A \rightarrow (B \rightarrow A))$ expanded to $(A \rightarrow (B \rightarrow A))$
axiom2 $((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))$ expanded to $((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))$
axiom3 $(((\neg B) \rightarrow (\neg A)) \rightarrow (((\neg B) \rightarrow A) \rightarrow B))$ expanded to $(((\neg B) \rightarrow (\neg A)) \rightarrow (((\neg B) \rightarrow A) \rightarrow B))$

3.3 Theorems

theorem1: $(((\neg a) \rightarrow a) \rightarrow a)$ **expanded to** $(((\neg a) \rightarrow a) \rightarrow a)$
1. $(((\neg B) \rightarrow (\neg A)) \rightarrow (((\neg B) \rightarrow A) \rightarrow B))$ add proposition axiom3
2. $(((\neg B) \rightarrow (\neg a)) \rightarrow (((\neg B) \rightarrow a) \rightarrow B))$ substitute A with a in 1
3. $(((\neg a) \rightarrow (\neg a)) \rightarrow (((\neg a) \rightarrow a) \rightarrow a))$ substitute B with a in 2
4. $((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))$ add proposition axiom2
5. $(((\neg a) \rightarrow (B \rightarrow C)) \rightarrow (((\neg a) \rightarrow B) \rightarrow ((\neg a) \rightarrow C)))$ substitute A with $(\neg a)$ in 4
6. $(((\neg a) \rightarrow (B \rightarrow (\neg a))) \rightarrow (((\neg a) \rightarrow B) \rightarrow ((\neg a) \rightarrow (\neg a))))$ substitute C with $(\neg a)$ in 5
7. $(((\neg a) \rightarrow ((B \rightarrow (\neg a)) \rightarrow (\neg a))) \rightarrow (((\neg a) \rightarrow (B \rightarrow (\neg a))) \rightarrow ((\neg a) \rightarrow (\neg a))))$ substitute B with $(B \rightarrow (\neg a))$ in 6
8. $(A \rightarrow (B \rightarrow A))$ add proposition axiom1
9. $((\neg a) \rightarrow (B \rightarrow (\neg a)))$ substitute A with $(\neg a)$ in 8

10. $((\neg a) \to ((B \to (\neg a)) \to (\neg a)))$ substitute B with $(B \to (\neg a))$ in 9
11. $(((\neg a) \to (B \to (\neg a))) \to ((\neg a) \to (\neg a)))$ modus ponens 7 and 10
12. $(A \to (B \to A))$ add proposition axiom1
13. $((\neg a) \to (B \to (\neg a)))$ substitute A with $(\neg a)$ in 12
14. $((\neg a) \to (\neg a))$ modus ponens 11 and 13
15. $(((\neg a) \to a) \to a)$ modus ponens 3 and 14
16. $(((\neg a) \to a) \to a)$ initial proposition

theorem2: $(a \to (a \vee a))$ **expanded to** $(a \to ((\neg a) \to a))$
1. $(A \to (B \to A))$ add proposition axiom1
2. $(a \to (B \to a))$ substitute A with a in 1
3. $(a \to ((\neg a) \to a))$ substitute B with $(\neg a)$ in 2

4 Kleene

4.1 Definitions

not $(\neg A)$ expanded to $(\neg A)$
or $(A \vee B)$ expanded to $(A \vee B)$
and $(A \wedge B)$ expanded to $(A \wedge B)$
impl $(A \to B)$ expanded to $(A \to B)$
equ $((A \to B) \wedge (B \to A))$ expanded to $((A \to B) \wedge (B \to A))$

4.2 Axioms

axiom1 $(A \to (B \to A))$ expanded to $(A \to (B \to A))$
axiom2 $((A \to (B \to C)) \to ((A \to B) \to (A \to C)))$ expanded to $((A \to (B \to C)) \to ((A \to B) \to (A \to C)))$
axiom3 $((A \vee B) \to A)$ expanded to $((A \vee B) \to A)$
axiom4 $((A \vee B) \to B)$ expanded to $((A \vee B) \to B)$
axiom5 $(A \to (B \to (A \wedge B)))$ expanded to $(A \to (B \to (A \wedge B)))$
axiom6 $(A \to (A \vee B))$ expanded to $(A \to (A \vee B))$
axiom7 $(B \to (A \vee B))$ expanded to $(B \to (A \vee B))$
axiom8 $((A \to C) \to ((B \to C) \to ((A \vee B) \to C)))$ expanded to $((A \to C) \to ((B \to C) \to ((A \vee B) \to C)))$
axiom9 $((A \to B) \to ((A \to (\neg B)) \to (\neg A)))$ expanded to $((A \to B) \to ((A \to (\neg B)) \to (\neg A)))$
axiom10 $((\neg(\neg A)) \to A)$ expanded to $((\neg(\neg A)) \to A)$

4.3 Theorems

theorem1: $((a \to (\neg a)) \to (\neg a))$ **expanded to** $((a \to (\neg a)) \to (\neg a))$
1. $((A \to B) \to ((A \to (\neg B)) \to (\neg A)))$ add proposition axiom9
2. $((a \to B) \to ((a \to (\neg B)) \to (\neg a)))$ substitute A with a in 1
3. $((a \to a) \to ((a \to (\neg a)) \to (\neg a)))$ substitute B with a in 2
4. $((A \to (B \to C)) \to ((A \to B) \to (A \to C)))$ add proposition axiom2
5. $((a \to (B \to C)) \to ((a \to B) \to (a \to C)))$ substitute A with a in 4

6. $((a \to (B \to a)) \to ((a \to B) \to (a \to a)))$ substitute C with a in 5
7. $((a \to ((A \vee a) \to a)) \to ((a \to (A \vee a)) \to (a \to a)))$ substitute B with $(A \vee a)$ in 6
8. $(A \to (B \to A))$ add proposition axiom1
9. $(a \to (B \to a))$ substitute A with a in 8
10. $(a \to ((A \vee a) \to a))$ substitute B with $(A \vee a)$ in 9
11. $((a \to (A \vee a)) \to (a \to a))$ modus ponens 7 and 10
12. $(B \to (A \vee B))$ add proposition axiom7
13. $(a \to (A \vee a))$ substitute B with a in 12
14. $(a \to a)$ modus ponens 11 and 13
15. $((a \to (\neg a)) \to (\neg a))$ modus ponens 3 and 14
16. $((a \to (\neg a)) \to (\neg a))$ initial proposition

theorem2: $(a \to (a \vee b))$ **expanded to** $(a \to (a \vee b))$
1. $(A \to (A \vee B))$ add proposition axiom6
2. $(a \to (a \vee B))$ substitute A with a in 1
3. $(a \to (a \vee b))$ substitute B with b in 2

theorem3: $(a \to (a \vee a))$ **expanded to** $(a \to (a \vee a))$
1. $(A \to (A \vee B))$ add proposition axiom6
2. $(a \to (a \vee B))$ substitute A with a in 1
3. $(a \to (a \vee a))$ substitute B with a in 2

5 Rasiowa and Sikorski

5.1 Definitions

not $(\neg A)$ expanded to $(\neg A)$
or $(A \vee B)$ expanded to $(A \vee B)$
and $(A \wedge B)$ expanded to $(A \wedge B)$
impl $(A \to B)$ expanded to $(A \to B)$
equ $((A \to B) \wedge (B \to A))$ expanded to $((A \to B) \wedge (B \to A))$

5.2 Axioms

axiom1 $((A \to B) \to ((B \to C) \to (A \to C)))$ expanded to $((A \to B) \to ((B \to C) \to (A \to C)))$
axiom2 $(A \to (A \vee B))$ expanded to $(A \to (A \vee B))$
axiom3 $(B \to (A \vee B))$ expanded to $(B \to (A \vee B))$
axiom4 $((A \to C) \to ((B \to C) \to ((A \vee B) \to C)))$ expanded to $((A \to C) \to ((B \to C) \to ((A \vee B) \to C)))$
axiom5 $((A \wedge B) \to A)$ expanded to $((A \wedge B) \to A)$
axiom6 $((A \wedge B) \to B)$ expanded to $((A \wedge B) \to B)$
axiom7 $((C \to A) \to ((C \to B) \to (C \to (A \wedge B))))$ expanded to $((C \to A) \to ((C \to B) \to (C \to (A \wedge B))))$
axiom8 $((A \to (B \to C)) \to ((A \wedge B) \to C))$ expanded to $((A \to (B \to C)) \to ((A \wedge B) \to C))$

axiom9 $(((A \wedge B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C)))$ expanded to $(((A \wedge B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C)))$
axiom10 $((A \vee (\neg A)) \rightarrow B)$ expanded to $((A \vee (\neg A)) \rightarrow B)$
axiom11 $((A \rightarrow (A \wedge (\neg A))) \rightarrow (\neg A))$ expanded to $((A \rightarrow (A \wedge (\neg A))) \rightarrow (\neg A))$
axiom12 $(A \vee (\neg A))$ expanded to $(A \vee (\neg A))$

5.3 Theorems

theorem1: $((a \rightarrow (\neg a)) \rightarrow (\neg a))$ **expanded to** $((a \rightarrow (\neg a)) \rightarrow (\neg a))$
1. $((A \vee (\neg A)) \rightarrow B)$ add proposition axiom10
2. $((A \vee (\neg A)) \rightarrow ((a \rightarrow (\neg a)) \rightarrow (\neg a)))$ substitute B with $((a \rightarrow (\neg a)) \rightarrow (\neg a))$ in 1
3. $(A \vee (\neg A))$ add proposition axiom12
4. $((a \rightarrow (\neg a)) \rightarrow (\neg a))$ modus ponens 2 and 3
5. $((a \rightarrow (\neg a)) \rightarrow (\neg a))$ initial proposition

theorem2: $(a \rightarrow (a \vee b))$ **expanded to** $(a \rightarrow (a \vee b))$
1. $(A \rightarrow (A \vee B))$ add proposition axiom2
2. $(a \rightarrow (a \vee B))$ substitute A with a in 1
3. $(a \rightarrow (a \vee b))$ substitute B with b in 2

theorem3: $((a \rightarrow b) \rightarrow ((b \vee c) \rightarrow (a \vee c)))$ **expanded to** $((a \rightarrow b) \rightarrow ((b \vee c) \rightarrow (a \vee c)))$
1. $((A \vee (\neg A)) \rightarrow B)$ add proposition axiom10
2. $((A \vee (\neg A)) \rightarrow ((a \rightarrow b) \rightarrow ((b \vee c) \rightarrow (a \vee c))))$ substitute B with $((a \rightarrow b) \rightarrow ((b \vee c) \rightarrow (a \vee c)))$ in 1
3. $(A \vee (\neg A))$ add proposition axiom12
4. $((a \rightarrow b) \rightarrow ((b \vee c) \rightarrow (a \vee c)))$ modus ponens 2 and 3
5. $((a \rightarrow b) \rightarrow ((b \vee c) \rightarrow (a \vee c)))$ initial proposition

theorem4: $(a \rightarrow (a \vee a))$ **expanded to** $(a \rightarrow (a \vee a))$
1. $(A \rightarrow (A \vee B))$ add proposition axiom2
2. $(a \rightarrow (a \vee B))$ substitute A with a in 1
3. $(a \rightarrow (a \vee a))$ substitute B with a in 2

theorem5: $(a \rightarrow (\neg(\neg a)))$ **expanded to** $(a \rightarrow (\neg(\neg a)))$
1. $((A \vee (\neg A)) \rightarrow B)$ add proposition axiom10
2. $((A \vee (\neg A)) \rightarrow (a \rightarrow (\neg(\neg a))))$ substitute B with $(a \rightarrow (\neg(\neg a)))$ in 1
3. $(A \vee (\neg A))$ add proposition axiom12
4. $(a \rightarrow (\neg(\neg a)))$ modus ponens 2 and 3
5. $(a \rightarrow (\neg(\neg a)))$ initial proposition

theorem6: $(((\neg a) \rightarrow b) \rightarrow ((\neg b) \rightarrow a))$ **expanded to** $(((\neg a) \rightarrow b) \rightarrow ((\neg b) \rightarrow a))$
1. $((A \vee (\neg A)) \rightarrow B)$ add proposition axiom10
2. $((A \vee (\neg A)) \rightarrow (((\neg a) \rightarrow b) \rightarrow ((\neg b) \rightarrow a)))$ substitute B with $(((\neg a) \rightarrow b) \rightarrow ((\neg b) \rightarrow a))$ in 1
3. $(A \vee (\neg A))$ add proposition axiom12
4. $(((\neg a) \rightarrow b) \rightarrow ((\neg b) \rightarrow a))$ modus ponens 2 and 3

5. $(((\neg a) \rightarrow b) \rightarrow ((\neg b) \rightarrow a))$ initial proposition

theorem7: $((a \rightarrow b) \rightarrow ((\neg b) \rightarrow (\neg a)))$ **expanded to** $((a \rightarrow b) \rightarrow ((\neg b) \rightarrow (\neg a)))$
1. $((A \vee (\neg A)) \rightarrow B)$ add proposition axiom10
2. $((A \vee (\neg A)) \rightarrow ((a \rightarrow b) \rightarrow ((\neg b) \rightarrow (\neg a))))$ substitute B with $((a \rightarrow b) \rightarrow ((\neg b) \rightarrow (\neg a)))$ in 1
3. $(A \vee (\neg A))$ add proposition axiom12
4. $((a \rightarrow b) \rightarrow ((\neg b) \rightarrow (\neg a)))$ modus ponens 2 and 3
5. $((a \rightarrow b) \rightarrow ((\neg b) \rightarrow (\neg a)))$ initial proposition

theorem8: $(((\neg a) \rightarrow (\neg b)) \rightarrow (b \rightarrow a))$ **expanded to** $(((\neg a) \rightarrow (\neg b)) \rightarrow (b \rightarrow a))$
1. $((A \vee (\neg A)) \rightarrow B)$ add proposition axiom10
2. $((A \vee (\neg A)) \rightarrow (((\neg a) \rightarrow (\neg b)) \rightarrow (b \rightarrow a)))$ substitute B with $(((\neg a) \rightarrow (\neg b)) \rightarrow (b \rightarrow a))$ in 1
3. $(A \vee (\neg A))$ add proposition axiom12
4. $(((\neg a) \rightarrow (\neg b)) \rightarrow (b \rightarrow a))$ modus ponens 2 and 3
5. $(((\neg a) \rightarrow (\neg b)) \rightarrow (b \rightarrow a))$ initial proposition

theorem9: $((a \vee (b \vee c)) \rightarrow ((a \vee b) \vee c))$ **expanded to** $((a \vee (b \vee c)) \rightarrow ((a \vee b) \vee c))$
1. $((A \vee (\neg A)) \rightarrow B)$ add proposition axiom10
2. $((A \vee (\neg A)) \rightarrow ((a \vee (b \vee c)) \rightarrow ((a \vee b) \vee c)))$ substitute B with $((a \vee (b \vee c)) \rightarrow ((a \vee b) \vee c))$ in 1
3. $(A \vee (\neg A))$ add proposition axiom12
4. $((a \vee (b \vee c)) \rightarrow ((a \vee b) \vee c))$ modus ponens 2 and 3
5. $((a \vee (b \vee c)) \rightarrow ((a \vee b) \vee c))$ initial proposition

theorem10: $((\neg (a \vee b)) \rightarrow (\neg a))$ **expanded to** $((\neg (a \vee b)) \rightarrow (\neg a))$
1. $((A \rightarrow B) \rightarrow ((\neg B) \rightarrow (\neg A)))$ add proposition theorem7
2. $((a \rightarrow B) \rightarrow ((\neg B) \rightarrow (\neg a)))$ substitute A with a in 1
3. $((a \rightarrow (a \vee b)) \rightarrow ((\neg (a \vee b)) \rightarrow (\neg a)))$ substitute B with $(a \vee b)$ in 2
4. $(A \rightarrow (A \vee B))$ add proposition axiom2
5. $(a \rightarrow (a \vee B))$ substitute A with a in 4
6. $(a \rightarrow (a \vee b))$ substitute B with b in 5
7. $((\neg (a \vee b)) \rightarrow (\neg a))$ modus ponens 3 and 6
8. $((\neg (a \vee b)) \rightarrow (\neg a))$ initial proposition

6 Modal

6.1 Definitions

not $(\neg A)$ expanded to $(\neg A)$
box $(\Box A)$ expanded to $(\Box A)$
or $(A \vee B)$ expanded to $(A \vee B)$
and $(A \wedge B)$ expanded to $(A \wedge B)$
impl $(A \rightarrow B)$ expanded to $(A \rightarrow B)$
equ $((A \rightarrow B) \wedge (B \rightarrow A))$ expanded to $((A \rightarrow B) \wedge (B \rightarrow A))$

notbox $(\neg(\Box(\neg A)))$ expanded to $(\neg(\Box(\neg A)))$

6.2 Axioms

axiom1 $((A \to B) \to ((B \to C) \to (A \to C)))$ expanded to $((A \to B) \to ((B \to C) \to (A \to C)))$
axiom2 $(A \to (A \vee B))$ expanded to $(A \to (A \vee B))$
axiom3 $(B \to (A \vee B))$ expanded to $(B \to (A \vee B))$
axiom4 $((A \to C) \to ((B \to C) \to ((A \vee B) \to C)))$ expanded to $((A \to C) \to ((B \to C) \to ((A \vee B) \to C)))$
axiom5 $((A \wedge B) \to A)$ expanded to $((A \wedge B) \to A)$
axiom6 $((A \wedge B) \to B)$ expanded to $((A \wedge B) \to B)$
axiom7 $((C \to A) \to ((C \to B) \to (C \to (A \wedge B))))$ expanded to $((C \to A) \to ((C \to B) \to (C \to (A \wedge B))))$
axiom8 $((A \to (B \to C)) \to ((A \wedge B) \to C))$ expanded to $((A \to (B \to C)) \to ((A \wedge B) \to C))$
axiom9 $(((A \wedge B) \to C) \to (A \to (B \to C)))$ expanded to $(((A \wedge B) \to C) \to (A \to (B \to C)))$
axiom10 $((A \vee (\neg A)) \to B)$ expanded to $((A \vee (\neg A)) \to B)$
axiom11 $((A \to (A \wedge (\neg A))) \to (\neg A))$ expanded to $((A \to (A \wedge (\neg A))) \to (\neg A))$
axiom12 $(A \vee (\neg A))$ expanded to $(A \vee (\neg A))$
axiom13 $(((\Box A) \wedge (\Box B)) \to (\Box(A \wedge B)))$ expanded to $(((\Box A) \wedge (\Box B)) \to (\Box(A \wedge B)))$
axiom14 $((\Box A) \to A)$ expanded to $((\Box A) \to A)$
axiom15 $((\Box A) \to (\Box(\Box A)))$ expanded to $((\Box A) \to (\Box(\Box A)))$
axiom16 $(\Box(A \vee (\neg A)))$ expanded to $(\Box(A \vee (\neg A)))$

6.3 Theorems

theorem1: $(((\Diamond a) \to (\Box b)) \to (a \to b))$ **expanded to** $(((\neg(\Box(\neg a))) \to (\Box b)) \to (a \to b))$
1. $((A \vee (\neg A)) \to B)$ add proposition axiom10
2. $((A \vee (\neg A)) \to (((\neg(\Box(\neg a))) \to (\Box b)) \to (a \to b)))$ substitute B with $(((\neg(\Box(\neg a))) \to (\Box b)) \to (a \to b))$ in 1
3. $(A \vee (\neg A))$ add proposition axiom12
4. $(((\neg(\Box(\neg a))) \to (\Box b)) \to (a \to b))$ modus ponens 2 and 3
5. $(((\neg(\Box(\neg a))) \to (\Box b)) \to (a \to b))$ initial proposition

theorem2: $(a \to (\Diamond a))$ **expanded to** $(a \to (\neg(\Box(\neg a))))$
1. $((A \vee (\neg A)) \to B)$ add proposition axiom10
2. $((A \vee (\neg A)) \to (a \to (\neg(\Box(\neg a)))))$ substitute B with $(a \to (\neg(\Box(\neg a))))$ in 1
3. $(A \vee (\neg A))$ add proposition axiom12
4. $(a \to (\neg(\Box(\neg a))))$ modus ponens 2 and 3
5. $(a \to (\neg(\Box(\neg a))))$ initial proposition

theorem3: $((\Box a) \to (\Diamond a))$ **expanded to** $((\Box a) \to (\neg(\Box(\neg a))))$
1. $((A \vee (\neg A)) \to B)$ add proposition axiom10
2. $((A \vee (\neg A)) \to ((\Box a) \to (\neg(\Box(\neg a)))))$ substitute B with $((\Box a) \to (\neg(\Box(\neg a))))$

in 1
3. $(A \vee (\neg A))$ add proposition axiom12
4. $((\Box a) \rightarrow (\neg(\Box(\neg a))))$ modus ponens 2 and 3
5. $((\Box a) \rightarrow (\neg(\Box(\neg a))))$ initial proposition

theorem4: $((\Box(a \wedge b)) \rightarrow ((\Box a) \wedge (\Box b)))$ **expanded to** $((\Box(a \wedge b)) \rightarrow ((\Box a) \wedge (\Box b)))$
1. $((A \vee (\neg A)) \rightarrow B)$ add proposition axiom10
2. $((A \vee (\neg A)) \rightarrow ((\Box(a \wedge b)) \rightarrow ((\Box a) \wedge (\Box b))))$ substitute B with $((\Box(a \wedge b)) \rightarrow ((\Box a) \wedge (\Box b)))$ in 1
3. $(A \vee (\neg A))$ add proposition axiom12
4. $((\Box(a \wedge b)) \rightarrow ((\Box a) \wedge (\Box b)))$ modus ponens 2 and 3
5. $((\Box(a \wedge b)) \rightarrow ((\Box a) \wedge (\Box b)))$ initial proposition

theorem5: $((\Diamond(a \wedge b)) \rightarrow ((\Diamond a) \wedge (\Diamond b)))$ **expanded to** $((\neg(\Box(\neg(a \wedge b)))) \rightarrow ((\neg(\Box(\neg a))) \wedge (\neg(\Box(\neg b)))))$
1. $((A \vee (\neg A)) \rightarrow B)$ add proposition axiom10
2. $((A \vee (\neg A)) \rightarrow ((\neg(\Box(\neg(a \wedge b)))) \rightarrow ((\neg(\Box(\neg a))) \wedge (\neg(\Box(\neg b))))))$ substitute B with $((\neg(\Box(\neg(a \wedge b)))) \rightarrow ((\neg(\Box(\neg a))) \wedge (\neg(\Box(\neg b)))))$ in 1
3. $(A \vee (\neg A))$ add proposition axiom12
4. $((\neg(\Box(\neg(a \wedge b)))) \rightarrow ((\neg(\Box(\neg a))) \wedge (\neg(\Box(\neg b)))))$ modus ponens 2 and 3
5. $((\neg(\Box(\neg(a \wedge b)))) \rightarrow ((\neg(\Box(\neg a))) \wedge (\neg(\Box(\neg b)))))$ initial proposition

theorem6: $((\Diamond(a \vee b)) \rightarrow (\Diamond a))$ **expanded to** $((\neg(\Box(\neg(a \vee b)))) \rightarrow (\neg(\Box(\neg a))))$
1. $((A \vee (\neg A)) \rightarrow B)$ add proposition axiom10
2. $((A \vee (\neg A)) \rightarrow ((\neg(\Box(\neg(a \vee b)))) \rightarrow (\neg(\Box(\neg a)))))$ substitute B with $((\neg(\Box(\neg(a \vee b)))) \rightarrow (\neg(\Box(\neg a))))$ in 1
3. $(A \vee (\neg A))$ add proposition axiom12
4. $((\neg(\Box(\neg(a \vee b)))) \rightarrow (\neg(\Box(\neg a))))$ modus ponens 2 and 3
5. $((\neg(\Box(\neg(a \vee b)))) \rightarrow (\neg(\Box(\neg a))))$ initial proposition

theorem7: $(((\Box a) \vee (\Box b)) \rightarrow (\Box(a \vee b)))$ **expanded to** $(((\Box a) \vee (\Box b)) \rightarrow (\Box(a \vee b)))$
1. $((A \vee (\neg A)) \rightarrow B)$ add proposition axiom10
2. $((A \vee (\neg A)) \rightarrow (((\Box a) \vee (\Box b)) \rightarrow (\Box(a \vee b))))$ substitute B with $(((\Box a) \vee (\Box b)) \rightarrow (\Box(a \vee b)))$ in 1
3. $(A \vee (\neg A))$ add proposition axiom12
4. $(((\Box a) \vee (\Box b)) \rightarrow (\Box(a \vee b)))$ modus ponens 2 and 3
5. $(((\Box a) \vee (\Box b)) \rightarrow (\Box(a \vee b)))$ initial proposition